A STUDY OF TEMPERATURE WAVES BASED ON THE GENERALIZED HEAT-TRANSFER EQUATION

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A hyperbolic-type thermal conductivity equation is considered for a medium with variable thermophysical parameters. New classes of temperature fields are obtained, which permit an accurate analytical description.

1. One-Dimensional Temperature Waves. In a plane one-dimensional region the temperature field is defined by the generalized heat-transfer equation [1-3]

$$cT_t + c\gamma T_{tt} = (\lambda T_x)_x + q(T,\ x,\ t). \tag{1}$$

Here and below the use of independent variables as subscripts will denote partial differentiation.

We will consider the temperature interval $T \in [T_1,\ T_2]$, in which the following formulas will describe the thermophysical properties of the medium:

$$c = c_0 + c_1 T,\ c_0,\ c_1 \text{— const};\ \lambda = \lambda_0 + \lambda_1 T,\ \lambda_0,\ \lambda_1 \text{— const}; \tag{2}$$

$$c\gamma = \varkappa_0 + \varkappa_1 T,\ \varkappa_0 = \varkappa_0(t),\ \varkappa_1 = \varkappa_1(t). \tag{3}$$

In particular, we are interested in the case where the heat-transfer relaxation time $\gamma = \gamma(t)$, so that

$$\varkappa_0 = c_0\gamma(t),\ \varkappa_1 = c_1\gamma(t). \tag{4}$$

Let there exist an immobile medium, heat transfer in which obeys Eqs. (1)-(3). The temperature of the medium is constant and equal to T_0. We take the energy liberation law in the form

$$q = q_0(t)(T - T_0)^n,\ n \geqslant 1. \tag{5}$$

Problem 1: at the initial moment in time

$$T(x,\ t)|_{x=0,\ t=0} = T_0 \equiv \text{const},$$

and at $t > 0$ the temperature on the boundary $x = 0$ changes by a law $T_b(t)$, $T_b(0) = T_0$, which is defined with a certain arbitrariness in the course of constructing the solution. In the medium there propagates a temperature wave $x = x_f(t)$. It is required to find the temperature regime at $x \in [0,\ x_f]$.

Another variant is possible, viz., Problem 2: the left boundary of the medium is considered mobile, $x = x_b(t)$, and on this boundary the temperature derivative

$$\left(\frac{\partial T}{\partial x}\right)_{x=x_b(t)} = r_b(t) \tag{6}$$

is specified.

The algorithm to be used for solving the problems is based on the method of constructing a special functional series [4].

We apply a Legendre transform to Eq. (1) and transform from the plane (x, t) to the plane of the new independent variables (r, t):

$$\Theta(r,\ t) = xT_x - T + Mt,\ M \equiv \text{const}, \tag{7}$$

$$r = T_x,\ x = \Theta_r,\ \frac{D(x,\ t)}{D(r,\ t)} = \Theta_{rr} \neq 0.$$

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This gives the following nonlinear equation:

$$(c_0 + c_1 T)\Theta_{rr}(M - \Theta_t) + (\varkappa_0 + \varkappa_1 T)(\Theta_{rt}^2 - \Theta_{rr}\Theta_{tt}) = \lambda_0 + \lambda_1 T + \Theta_{rr}[\lambda_1 r^2 + q_0 (T - T_0)^n], \tag{8}$$

$$T(r, t) = Mt + r\Theta_r - \Theta.$$

Equation (8) has the characteristic r = 0, so that it is convenient to take the solution in the form of the functional series

$$\Theta(r, t) = \sum_{k=0}^{\infty} a^{(k)}(t) r^k. \tag{9}$$

In the region ahead of the temperature wave front, which is a line of weak discontinuity [5], r = 0, and for the first two terms of expansion (9) we obtain

$$a^{(0)}(t) = Mt - T_0,\ T_f(t) = T(r, t)|_{r=0} \equiv T_0,\ x_f(t) = (\Theta_r)_{r=0},\ x_f = a^{(1)}(t). \tag{10}$$

The calculations related to substitution of series (9) in Eq. (8) give us the equation (n = 1)

$$\sum_{k=0}^{\infty}\Big[M(c_0 + c_1 Mt)(k+2)(k+1)a^{(k+2)} - \sum_{i=0}^{k}\{(c_0 + c_1 Mt)(i+2)(i+ \tag{11}$$

$$+1)a^{(i+2)}a_t^{(k-i)} - c_1 M(i+2)(i+1)a^{(i+2)}a^{(k-i)} + (\varkappa_0 + \varkappa_1 Mt)((i+1)a_t^{(i+1)}(k-i+1)a_t^{(k-i+1)} -$$

$$-(i+2)(i+1)a^{(i+2)}a_{tt}^{(k-i)})\} + \sum_{j=0}^{k}\Big\{c_1 a^{(k-j)} \sum_{i=0}^{j}(i+2)(i+1)a^{(i+2)}a_t^{(j-i)} - \varkappa_1(k-j+1)a_t^{(k-j+1)} \times$$

$$\times \sum_{i=0}^{j}(j-i+1)a^{(i)}a_t^{(j-i+1)} + \varkappa_1 a_{tt}^{(k-j)}\sum_{i=0}^{j}(j-i+2)\times$$

$$\times(j-i+1)a^{(i)}a^{(j-i+2)}\Big\}\Big] r^k + \sum_{k=0}^{\infty}\Big[\sum_{i=0}^{k} c_1 M(i+2)(i+1)\times$$

$$\times(k-i+1)a^{(i+2)}a^{(k-i+1)} + \sum_{j=0}^{k}\Big\{\varkappa_1(k-j+1)a_t^{(k-j+1)}\times$$

$$\times\sum_{i=0}^{j}(i+1)a^{(i+1)}(j-i+1)a_t^{(j-i+1)} - \varkappa_1 a_{tt}^{(k-j)}\sum_{i=0}^{j}(i+1)a^{(i+1)}\times$$

$$\times(j-i+2)(j-i+1)a^{(i+1)}a^{(j-i+2)} - c_1(k-j+1)a^{(k-j+1)}\times$$

$$\times\sum_{i=0}^{j}(i+2)(i+1)a^{(i+2)}a_t^{(j-i)}\Big\}\Big] r^{k+1} = \lambda_0 + \lambda_1 Mt + \sum_{k=0}^{\infty}\Big[q_0(Mt - T_0)(k+2)(k+1)a^{(k+2)} -$$

$$-q_0\sum_{i=0}^{k}(i+2)(i+1)a^{(i+2)}a^{(k-i)} - \lambda_1 a^{(k)}\Big] r^k +$$

$$+\sum_{k=0}^{\infty}\Big[\lambda_1(k+1)a^{(k+1)} + q_0\sum_{i=0}^{k}(i+2)(i+1)(k-i+1)a^{(i+2)}a^{(k-i+1)}\Big] r^{k+1} +$$

$$+\lambda_1\sum_{k=0}^{\infty}(k+2)(k+1)a^{(k+2)}r^{k+2}.$$

Comparing the terms with identical powers of the quantity r, we write equations for the coefficients of the series:

$$a_t^{(1)} = w_0(t),\ w_0(t) = w(T_0, t),\ w^2 = \frac{\lambda}{c\gamma}, \tag{12}$$

$$2\varkappa^0 w_0 a_t^{(2)} - B_0 a^{(2)} = 0, \tag{13}$$

$$6\varkappa^0 w_0 a_t^{(3)} - 6B_0 a^{(3)} - 2a^{(2)}a_t^{(2)}c(T_0) + 4\varkappa^0(a_t^{(2)})^2 - 2\varkappa^0 a^{(2)}a_{tt}^{(2)} + k_1 a^{(2)}w_0^2 - 3\lambda_1 a^{(2)} - 2q_0(a^{(2)})^2 = 0, \tag{14}$$

$$2\varkappa^0 w_0(k+2)a_t^{(k+2)} - [(k+2)^2 - (k+2)]B_0 a^{(k+2)} =$$

$$= F^{(k+2)}(t),\ k \geqslant 0;\ \varkappa = c\gamma,\ \varkappa^0 = \varkappa^0(t),\ B_0 = c(T_0)w_0(t) + \varkappa^0(t)w_{0t}. \tag{15}$$

Here the functions $F^{(k+2)}(t)$ contain coefficients of series (9) only with numbers smaller than (k + 2), i.e., Eq. (15) is a linear first-order equation. Performing the integration, we find

$$a^{(k+2)}(t)=\exp\left[-\int_0^t f(z)\,dz\right]\left(C^{(k+2)}+\int_0^t g(y)\quad\exp\left[\int_0^t f(z)\,dz\right]dy\right),\ k\geqslant 0, \tag{16}$$

$$f(t)=-\frac{B_0(k+1)}{2\varkappa^0 w_0},\ g(t)=\frac{F^{(k+2)}(t)}{2\varkappa^0 w_0(k+2)}\varkappa^0=\varkappa(T_0,\ t).$$

Thus, the coefficients of series (9) are defined in quadratures and each coefficient $a^{(k+2)}(t)$ contains one arbitrary constant $C^{(k+2)}$. The temperature field is characterized by the formulas

$$x(r,\ t)=\sum_{k=0}^{\infty}(k+1)a^{(k+1)}(t)\,r^k,\ T(r,\ t)=T_0+xr-\sum_{k=1}^{\infty}a^{(k)}(t)\,r^k.$$

The operations performed with the series are valid, since the convergence of series of the form of Eq. (9), which appear in the method of [4], was proved in [6].

From Eqs. (10), (12), we obtain the coordinate of the wave front

$$x_f(t)=\int_0^t w_0(z)\,dz,\ w_0=w(T_0,\ t). \tag{17}$$

The coordinate of the boundary x = 0 is defined in the new variables by the function $r = r_b(t)$ from the equation

$$x\,[r_b(t),\ t]=0,$$

while the boundary temperature $T_b(t) = T(r_b,\ t)$ is concretized by choice of the arbitrary constants $C^{(k+2)}$, $k = 0,\ 1,\ 2,\ \ldots,\ \infty$.

If the left-hand boundary is mobile (Problem 2), we proceed as follows: we specify the temperature derivative with respect to the spatial coordinate $r = r_b(t)$, while the law for boundary motion $x_b = x(r_b,\ t)$ can be varied by using the arbitrary constants $C^{(k+2)}$, $k \geqslant 0$.

We note that the dependence of the thermophysical parameters on temperature and the effect of heat absorption (liberation) is perceptible for the given class of solutions only beginning with third-order terms in series (9). If n > 1 in Eq. (5), then energy liberation affects only terms of even higher order.

It is known that in nonlinear media, localization of the region of temperature front propagation can occur. This is caused either by the character of the function $\lambda = \lambda(T - T_0)$ and the temperature regime at the boundary or by some special law of volume heat absorption. A detailed exposition of this question, together with a bibliography, may be found in [7].

Equation (17) shows that for the given type of temperature fields spatial localization of temperature perturbations

$$x_f(t)\leqslant x_*<\infty,\ t\in[0,\ \infty),\ T(x,\ t)\equiv T_0,\ |x|\geqslant x_*$$

can occur only because of the properties of the medium, which define the function $w_0 = w_0(t) < \infty$.

2. New Exact Solution of the Nonlinear Wave Heat-Transfer Equation. For the case of a high-intensity nonstationary heat-transfer process it is possible [8] that the term cT_t may be very small in comparison to $c\gamma T_{tt}$, and can thus be omitted.

We will accept this assumption, and further assume in Eq. (1) that $\lambda \equiv$ const, q = 0, and then write the nonlinear wave heat-transfer equation [8, 9]

$$T_{tt}=w^2T_{xx},\ w^2=\frac{\lambda}{c\gamma},\ w=w(T,\ T_x,\ T_t,\ x,\ t). \tag{18}$$

For the velocity of heat propagation, we assume a function $w = w(T_x,\ t)$.

Legendre transformation reduces Eq. (18) to a nonlinear Ampere-type equation

$$\Theta_{rr}\Theta_{tt}-\Theta_{rt}^2=-w^2(r,\ t), \qquad (19)$$

$$x=\Theta_r,\ r=T_x,\ T=xr-\Theta(r,\ t). \qquad (20)$$

An equation of the form of Eq. (19) is employed in nonstationary gasdynamics in the study of processes which are adiabatic but not isentropic (see [10] and the bibliography therein). We will make use of the available results of mathematical study of equations of the form of Eq. (9).

In [11] the intermediate integral method was used to prove that if

$$w^2=\frac{f^2(\alpha)}{(t+b)^4},\quad \alpha=\frac{r+a}{t+b}, \qquad (21)$$

the solution has a form

$$\Theta(r,\ t)=\pm\int f(\alpha)\,d\alpha+(t+b)\,\varphi(\alpha)+A,\ A\equiv\text{const}, \qquad (22)$$

while $f(\alpha)$, $\varphi(\alpha)$ are arbitrary functions. The signs ± refer to two different families of characteristics of Eq. (19). By choice of the function $f(\alpha)$ we have the possibility of varying the dependence of heat propagation velocity upon time and the derivative T_x in Eq. (21). The presence of the arbitrary function $\varphi(\alpha)$ permits examination of some boundary problems.

According to Eqs. (20), (22) we have

$$x(r,\ t)=\pm\frac{f(\alpha)}{t+b}+\varphi'(\alpha), \qquad (23)$$

$$T(r,\ t)=r\left(\varphi'\pm\frac{f}{t+b}\right)\mp\int f(\alpha)\,d\alpha-(t+b)\,\varphi-A. \qquad (24)$$

Example. Let $f(\alpha)=f_0\alpha^n$, $f_0\equiv\text{const}$, $n+1\neq 0$, while the temperature wave propagates in a medium with constant temperature $T_0\equiv\text{const}$. Then from the condition $T(r,\ t)|_{r=0}=T_0$ we find the function $\varphi(\alpha)$:

$$\varphi(\alpha)=-\alpha\mp\frac{f_0\alpha^{n+2}}{a(n+1)},\ A=a-T_0. \qquad (25)$$

The temperature field is represented in parametric form $x=x(r,\ t)$, $T=T(r,\ t)$ with Eqs. (23)-(25).

The coordinate of the wave front is as follows:

$$x_f(t)=-1+\left(\frac{b}{t+b}\right)^{n+1},\ n+1\neq 0,\ b>0,$$

$$x_f(t)\to -1,\ t\to\infty,\ n+1>0;\ x\in(-1,\ 0],$$

$$x_f(t)\to\infty,\ t\to\infty,\ n+1<0;\ x\in[0,\ \infty).$$

We see that in this example localization of temperature perturbations is produced by the choice of n, i.e., by the properties of the medium, Eq. (21).

Taking $n = 0$ for simplicity, we find that the value $x_b(t)\equiv 0$ in Problem 1 corresponds to

$$r_b(t)+a=\pm\frac{a}{2f_0}(t+b\mp f_0),$$

so that $T_b(t)=T(r_b,\ t)$ by Eq. (24).

It is simple to see that for Problem 2 one of the functions $x_b(t)$, $r_b(t)$ can be considered as specified beforehand; the remaining formulas remain unchanged.

It should be noted that the class of functions (21), characterizing the dependence $w(r, t)$ (or $\gamma(r,\ t)$)) and permitting solution of Eq. (19) with arbitrariness of one argument in one function, may be expanded. In fact, if, following [12], we take

$$w^2(r,\ t)=\frac{f^2(\alpha)}{(t+b)^4}+\sum_{k=4}^{2N+4}L_{\frac{k-4}{2}}(\alpha)\,z^{\frac{k-4}{2}},\ N\geqslant 0,$$

$$\alpha = \frac{r+a}{t+b}, \; z = (r+a)(t+b), \tag{26}$$

then the solution of Eqs. (19), (26) has the form

$$\Theta(r, t) = \Theta_0(r, t) + \sum_{k=4}^{\infty} \chi_{k-2}(\alpha) z^{\frac{k}{2}}, \tag{27}$$

while $\theta_0(r, t)$ is calculated from Eq. (22), and the functions are arbitrary.

The coefficients $\chi_{k-2}(\alpha)$ in Eq. (27) are found from the solution of the following first-order linear equations:

$$6\chi_2' f\alpha^2 + 12\chi_2\alpha(f + \alpha f') + L_0 = 0,$$

$$8\chi_3' f\alpha^2 + 20\chi_3\alpha(f + \alpha f') + 12\chi_2\varphi''\alpha^{\frac{3}{2}} + L_{\frac{1}{2}} = 0,$$

$$10\chi_4' f\alpha^2 + 30\chi_4\alpha(f + \alpha f') + 20\alpha^{\frac{3}{2}}\varphi''\chi_3 + L_1 = 0,$$

$$(m+3)\alpha[(m+4)\chi_{m+2}(\alpha f' + f) + 2\chi_{m+2}'\alpha f + (m+2)\chi_{m+1}\alpha^{\frac{1}{2}}\varphi''] + L_{\frac{m}{2}} +$$

$$+ \sum_{i=4}^{m}\Big[i(i+1)\alpha^2\chi_{i-2}\chi_{m+2-i}' + i(i-1)\alpha\chi_{i-2}\chi_{m+2-i}' +$$

$$+ i(i-1)\frac{(i-m-4)}{4}\chi_{i-2}\chi_{m+2-i} + (i^2 - mi - 2i - 1)\alpha^2\chi_{i-2}'\chi_{m+2-i}'\Big] = 0,$$

where $m \geqslant 0$; in this recursive formula the last sum is not equal to zero at $m \geqslant 4$. Integration of these equations presents no difficulties, and each coefficient $\chi_{k-2}(\alpha)$ is defined with arbitrariness in one integration constant C_{k-2}, $k \geqslant 4$.

Consequently, the expression for the temperature T(r, t) will contain an arbitrary function $\varphi(\alpha)$ and arbitrary constants C_{k-2}, k = 4, 5, ..., ∞, while function w(r, t) is determined by the choice of 2N + 2 arbitrary functions with argument α; $N \geqslant 0$ is an integer.

The local convergence of the series appearing in solution (27) may be proved by the method of Weierstrass–Kovalevskaya majority functions [13]. We then find that if $\chi_2(\alpha)$ is an analytical function and the boundary of the region under study and the conditions thereon are also analytical, then

$$\Theta_1(\alpha, z) = \sum_{k=4}^{\infty} \chi_{k-2}(\alpha) z^{\frac{k}{2}}$$

is analytic with respect to the argument z in some vicinity of the chosen value $z = z^{\circ}$. The technique of the proof is analogous to [14].

In conclusion, we formulate the following analogy. In one-dimensional nonstationary gasdynamics, for adiabatic processes simple Riemann waves are known, and their generalization is nonisentropic simple waves [10]. Correspondingly, in the heat-transfer theory simple waves occur at $w = w(T_x)$ [9], while their generalization to the case $w = w(T_x, t)$ is represented by Eqs. (21), (22). The generalization of both simple gasdynamic waves and simple thermal waves is then based on the intermediate integral method for the Monzh–Ampere equation (19).

NOTATION

T, temperature; x, Cartesian coordinate; t, time; c, specific heat; λ, thermal conductivity of the medium; γ, heat-transfer relaxation period; w, heat propagation velocity; q, volume energy source; Θ, auxiliary function; r = ∂T/∂x. Subscripts: f, value on wave front; b, on boundary; superscript in curved brackets, number of series term; independent variable used as subscript denotes partial differentiation; prime above a function sign, differentiation of a single-argument function.

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INVERSE BOUNDARY-VALUE PROBLEM OF HEAT CONDUCTION FOR A TWO-DIMENSIONAL DOMAIN

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An approximate solution of a two-dimensional inverse problem is constructed on the basis of a solution of the Cauchy problem obtained in the form of a series in the derivatives and the Tikhonov regularization method.

The thermal state of power equipment is determined to a considerable extent by the heat-transfer characteristics on the surface of the structure elements. These conditions can often be found only from the solution of the inverse boundary-value problems of heat conduction. Such one-dimensional problems have been studied sufficiently completely [1]. However, the one-dimensional model cannot yield confident results for nonuniform heat delivery and thickness of the structure element.

Let us examine the problem of determining the temperature and heat fluxes from a heat-delivering boundary $y = W(x)$, $(0 < W(x) < d)$ of the two-dimensional domain $D = \{(x, y): x \in [0, d], y \in [0, W(x)]\}$ by means of known temperature measures and the law of heat transfer to the opposite side, which is given by the line $y = 0$. We consider the thermophysical parameters constant.

Let the curve $y = W(x)$ have a continuous external normal $n(x)$ and at points defined by the mesh $\omega_x = \{x_0 < x_1 < \ldots < x_k\}$ on the boundary $y = 0$ let the temperature $t(x, y, \tau)$ be known at the times $\omega_\tau = \{\tau_0 < \tau_1 < \ldots < \tau_p\}$, i.e.,

$$t(x_i, 0, \tau_j) = f_{ij}. \quad (1)$$

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